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ON THE TERNARY QUADRATIC DIOPHANTINE EQUATION $x^2 + y^2 = z^2 + 13$

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ABSTRACT

This paper concerns the problem of obtaining non-zero distinct integer solutions to the given ternary quadratic diophantine equation by introducing linear transformations. The Ternary quadratic equation represented by x2 + y2= z2 +13is analyzed for its distinct integer solutions by using linear transformations and converting it to a binary quadratic equation. A few interesting relations among the solutions and expressions for special integers are given. Further, from the resulting solutions of the binary quadratic equation, we have obtained solutions of other choices of hyperbola, and parabola.

KEYWORDS: Ternary Quadratic, Binary Quadratic, Hyperbola, Parabola, Pell Equation, Nasty Numbers & Integral Solutions.

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Original Article

INTRODUCTION

A Binary quadratic equation of the form $y^2 = Dx^2 + 1$ where D is non-square positive integer has been studied by various mathematicians for its non-trivial integral solutions. When D takes different integral values In this communication, a Ternary quadratic equation represented by $x^2 + y^2 = z^2 + 13$is analyzed for its distinct integer solutions by converting it to a binary quadratic equation using linear transformations and infinitely many integer solutions are obtained. A few interesting properties among the solutions are obtained. Employing the solutions of the given equation, integer solutions to special hyperbolas and parabolas are obtained.

Consider the Ternary Quadratic Diophantine Equation

$$x^2 + y^2 = z^2 + 13 \quad (1)$$

Put $x=2\alpha$, $z=4\alpha$, equation (1) reduces to

$$y^2 = 12\alpha^2 + 13 \quad (2)$$

Equation (2) is a positive pell equation, one solution of (2) is $\alpha_0=1$, $y_0=5$

Corresponding pell equation is $y^2=12\alpha^2+1$ (3)

One solution of (3) is $\tilde{\alpha}_0=2$, $\tilde{y}_0=7$

General solution of equation(3) is

$\tilde{y}_n=\frac{1}{2} f_n,$

$\tilde{\alpha}_n=\frac{1}{2\sqrt{D}}g_n=\frac{1}{2\sqrt{12}}g_n$

Where $f_n=(\tilde{y}_0+\sqrt{D}\,\tilde{\alpha}_0)^{n+1}+(\tilde{y}_0-(\sqrt{D}\,\tilde{\alpha}_0)^{n+1}$

$g_n=(\tilde{y}_0+(\sqrt{D})\tilde{\alpha}_0)^{n+1}-(\tilde{y}_0-(\sqrt{D})\,\tilde{\alpha}_0)^{n+1}$

$$\left.\begin{aligned} &\text{so } f_n=(7+4\sqrt{3})^{n+1}+(7-4\sqrt{3})^{n+1}\\ &g_n=(7+4\sqrt{3})^{n+1}-(7-4\sqrt{3})^{n+1}\\ &\tilde{\alpha}_n=\frac{1}{2\sqrt{12}}[(7+4\sqrt{3})^{n+1}-(7-4\sqrt{3})^{n+1}]\end{aligned}\right\}\quad n=-1,0,1,2,3.. \qquad (4)$$

$\tilde{y}_n=\frac{1}{2}[(7+4\sqrt{3})^{n+1}+(7-4\sqrt{3})^{n+1}]$

To obtain the sequence of solutions of (1), we apply a lemma known as

Brahmagupta lemma stated as follows:

Brahmagupta lemma If (x_0, y_0) and (x_1, y_1) represent the solutions of the pell equations $y^2 = Dx^2 + k_1$ and $y^2 = Dx^2 + k_2$ $(D > 0\ \ and\ \ square\ \ free)$ respectively, then $(x_0y_1 + y_0x_1,\ y_0y_1 + Dx_0x_1)$ represents the solution of the pell equation $y^2 = Dx^2 + k_1k_2$

Applying Brahmagupta lemma between (α_0 y_0)& ($\tilde{\alpha}_n$ $\tilde{y}_n$) of (2) and (3) the other integer solutions of (1) are given by

$\alpha_{n+1}=\alpha_0\,\tilde{y}_n+y_0\tilde{\alpha}_n=\frac{1}{2}f_n+\frac{5}{2\sqrt{12}}g_n$ (5)

$y_{n+1}=y_0\,\tilde{y}_n+D\,\alpha_0\,\tilde{\alpha}_n=\frac{5}{2}f_n+\frac{12}{2\sqrt{12}}g_n$ (6)

$\frac{5}{2}f_n+\frac{\sqrt{12}}{2}g_n$

from equation (2), x=2α, z=4α

$$\begin{aligned} x_{n+1}&=f_n+\frac{5}{\sqrt{12}}g_n\\ z_{n+1}&=2f_n+\frac{5}{\sqrt{3}}g_n \qquad (7)\\ y_{n+1}&=\frac{5}{2}f_n+\sqrt{3}g_n\end{aligned}$$

equation (7) represents integer solutions to equation (1)

A few numerical examples are given in the following table

Table 1: Numerical Examples

n	x_{n+1}	y_{n+1}	z_{n+1}
-1	2	5	4
0	34	59	68
1	474	821	948
2	6602	11435	13204

3	91954	159269	183908

Recurrence Relations among the Solutions

$x_{n+3}=14\,x_{n+2}-x_{n+1}$

$y_{n+3}=14\,y_{n+2}-y_{n+1}$

$z_{n+3}=14\,z_{n+2}-z_{n+1}$

A Few Interesting Relations among the Solutions are given below

$x_{n+2}=7x_{n+1}-4y_{n+1}$

$x_{n+3}=56y_{n+1}+97x_{n+1}$

$y_{n+2}=7y_{n+1}+12x_{n+1}$

$y_{n+3}=97y_{n+1}+168x_{n+1}$

Definition: Nasty Number

A positive integer n is a Nasty number if $n=ab=cd$ and $a+b=c-d$ or $a-b=c+d$ where a, b, c, d are non-zero distinct positive integers(Bert Miller 1980).

Using the relation $f_n^2=f_{2n+1}+2$and solving for fn using any two equations among the solutions we get the following expressions as **Nasty number**

N_1: $12+\frac{6}{13}[10y_{2n+2}-12x_{2n+2}]$

N_2:$12+\frac{6}{26}[5x_{2n+3}-59x_{2n+2}]$

N_3:$12+\frac{6}{26}[59x_{2n+4}-821x_{2n+3}]$

N_4:$12+\frac{6}{13}[118y_{2n+3}-204x_{2n+3}]$

Each of the following Expressions Represents a Cubical Integer

Using the relation $f_n^3=f_{3n+2}+3f_n$and solving for fn using any two equations among the solutions we get the following expressions as cubical integer

$Q_1=\frac{1}{26}[5x_{3n+4}-59x_{3n+3}]+\frac{3}{26}[5x_{n+2}-59x_{n+1}]$

$Q_2=\frac{1}{13}[10y_{3n+3}-12x_{3n+3}]+\frac{3}{26}[10y_{n+1}-12x_{n+1}]$

$Q_3=\frac{1}{26}[59x_{3n+5}-821x_{3n+4}]+\frac{3}{26}[59x_{n+3}-821x_{n+2}]$

$Q_4=\frac{1}{13}[118y_{3n+4}-204x_{3n+4}]+\frac{3}{13}[118y_{n+2}-204x_{n+2}]$

Each of the following expressions represents a Biquadratic Integer

Using the relation $f_n^4=4f_n^2+f_{4n+3}-2$and solving for fn using any two equations among the solutions we get the following expressions as **Biquadratic Integer**

$B_1=4[2+\frac{1}{13}(10y_{2n+2}-12x_{2n+2})]-2+\frac{1}{13}[10y_{4n+4}-12x_{4n+4}]$

$B_1=6+\frac{4}{13}[(10y_{2n+2}-12x_{2n+2})]+\frac{1}{13}[(10y_{4n+4}-12x_{4n+4})]$

$B_2=6+\frac{1}{26}[5x_{2n+3}-59x_{2n+2}]+\frac{1}{26}[5x_{4n+5}-59x_{4n+4}]$

$B_3=6+\frac{1}{26}[59x_{2n+4}-821x_{2n+3}]+\frac{1}{26}[59x_{4n+6}-821x_{4n+5}]$

$B_4=6+\frac{1}{13}[118y_{2n+3}-204x_{2n+3}]+\frac{1}{13}[118y_{4n+5}-204x_{4n+5}]$

Each of the following expressions represent Parabolas:

Table 2: Parabolas

S.No	Parabolas	(X,Y)
1	$3Y^2 = 169(X-4)$	$(2+\frac{1}{26}[5x_{2n+3}-59x_{2n+2}], [17x_{n+1}-x_{n+2}])$
2	$12Y^2 = 169(X-4)$	$(2+\frac{1}{26}[5x_{2n+3}-59x_{2n+2}], [17x_{n+1}-x_{n+2}])$
3	$3Y^2 = 169(X-4)$	$(2+\frac{1}{26}[59x_{2n+4}-821x_{2n+3}], [237x_{n+2}-17x_{n+3}])$
4	$12Y^2 = 169(X-4)$	$(2+\frac{1}{13}[118y_{2n+3}-204x_{2n+3}], [59x_{n+2}-34y_{n+2}])$

Each of the following expressions represent Hyperbolas

Table 3: Hyperbolas

S.No	Hyperbolas	(X, Y)
1	$X^2 - Y^2 = 4$	$(\frac{1}{26}[5x_{n+2}-59x_{n+1}], \frac{1}{13}[17x_{n+1}-x_{n+2}])$
2	$X^2 - 12Y^2 = 4$	$(\frac{1}{13}[10y_{n+1}-12x_{n+1}], \frac{1}{13}[5x_{n+1}-2y_{n+1}]$
3	$X^2 - 3Y^2 = 4$	$(\frac{1}{26}(59x_{n+3}-821x_{n+2}), \frac{1}{13})(237x_{n+2}-17x_{n+3}))$
4	$x^2-12y^2=4$	$(\frac{1}{13}(118y_{n+2}-204x_{n+2}), \frac{1}{13}(59x_{n+2}-34y_{n+2}))$

Solution Set 2: Introducing the linear transformations

$x=3\alpha z=6\alpha$ the Diophantine equation

$x^2+y^2=z^2+13$ changes to

$$y^2=27\alpha^2+13 \quad (8)$$

which is the positive pell equation.

The smallest positive integer solution of (8) is

$\alpha_0=2$ $y_0=11$

To obtain the other solutions of (8), consider its corresponding positive pell equation represented by

$y^2=27\alpha^2+1$

One solution of (3) is $\tilde{\alpha}_0=5$, $\tilde{y}_0=26$

General solution of equation (8) can be found as in previous case

Solution Set 3: Introducing the linear transformations

$x=5\alpha$, $z=8\alpha$

the Diophantine equation

$x^2+y^2=z^2+13$ changes to

$$y^2=39\alpha^2+13 \tag{9}$$

which is the positive pell equation.

The smallest positive integer solution of (9) is

$\alpha_0=2$ $y_0=13$

To obtain the other solutions of (9), consider its corresponding positive pell equation represented by

$y^2=39\alpha^2+1$

One solution of (3) is $\tilde{\alpha}_0=4$, $\tilde{y}_0=25$

General solution of equation (9) can be found as in previous case

Solution **Set: 4**

Introducing the linear transformations

$$z=u+v\,,\ y=u-v \tag{10}$$

in (3.47), it leads to

$$x^{2=}4uv+13 \tag{11}$$

Choose u and v such that the R.H.S of (11) is a perfect square and from which the x-value is obtained. Substituting the corresponding values of u and v in (10), y and z values are obtained. This process is illustrated in the following Table: 4

Table 4: Illustrations

v	u	x	y	z
1	k^2+3k-1	k^2+3k	k^2+3k-2	$2k+3$
1	$4k^2+10k+3$	$4k^2+10k+4$	$4k^2+10k+2$	$4k+5$
3	$3k^{2+}5k+1$	$3k^{2+}5k+4$	$3k^{2+}5k-2$	$6k+5$
1	$4k^{2+}6k-1$	$4k^{2+}6k$	$4k^{2+}6k-2$	$4k+3$
1	$k^{2+}9k+17$	$k^{2+}9k+18$	$k^{2+}9k+16$	$2k+9$

CONCLUSIONS

In this paper, we presented infinitely many integer solutions to the Ternary quadratic equation represented by $x^2 + y^2= z^2 +13$ along with suitable relations between the solutions,The solution given by different methods or choices is unique in its own way and each system gives infinitely many solutions, but also not all the solutions. So there may be other choices of

transformations or methods to find some more infinite number of solutions Since Ternary Quadratic Diophantine equations are infinite, one may attempt to determine other integer solutions of the same or other equations of degree 2 as well as higher degree with suitable properties.

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